J. Math. Anal. Appl. 385 (2012) 264–277

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Local behavior of planar analytic vector fields via integrability ☆

María Jesús Álvarez [a], Antoni Ferragut [b,*]

[a] Departament de Matemàtiques i Informàtica, Universitat de les Illes Balears, Edifici Anselm Turmeda, Ctra. de Valldemossa, km 7.5, 07122 Palma, Illes Balears, Spain
[b] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Av. Diagonal, 647, 08028 Barcelona, Catalonia, Spain

A R T I C L E I N F O

Article history:
Received 22 December 2009
Available online 29 June 2011
Submitted by J. Mawhin

A.F. wants to dedicate this paper to the memory of his father

Keywords:
Planar analytic vector field
Generalized rational first integral
Blow-up

A B S T R A C T

We present an algorithm to study the local behavior of singular points of planar analytic vector fields having a first integral which is a quotient of analytic functions. The algorithm is based on the blow-up method. It emphasizes the curves passing through the singular points and avoids the computation of the desingularized systems. Vector fields having a rational first integral are a particular case.

1. Introduction

A *real planar analytic vector field* is a vector field defined on $\mathbb{R}^2$ of the form

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \tag{1}$$

where P and Q are coprime analytic functions. We refer to the vector field (1) or equivalently to its associated *planar analytic differential system*

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y). \tag{2}$$

Let $m = \min\{m_P, m_Q\}$ be the *multiplicity of the vector field* (1) *at the origin*, where m_P and m_Q are the multiplicities of P and Q at the origin, respectively.

The study of the topological behavior of the solutions of a planar differential system in a neighborhood of a singular point is one of the main unsolved problems in the qualitative theory of differential systems. Concerning the singular points having at least one eigenvalue different from zero, the problem is solved except for the center-focus case. Regarding the degenerate singular points, with both eigenvalues of the jacobian matrix at the point equal to zero, the situation is more complicated. The topology around a non-monodromic singular point can be much richer. The Andreev Theorem (see [3]) classifies the nilpotent singular points (degenerate singular points whose associated jacobian matrix is not identically zero) except the center-focus case. If the jacobian matrix is identically null the problem is open. In this case, the only possibility is studying each degenerate point case by case. The main technique which is used to perform the study of this kind of points

☆ The authors are partially supported by MTM2008-03437. A.F. is additionally partially supported by grants *Juan de la Cierva*, 2009SGR410 and MTM2009-14163-C02-02.

* Corresponding author.
E-mail addresses: chus.alvarez@uib.es (M.J. Álvarez), antoni.ferragut@upc.edu (A. Ferragut).

0022-247X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2011.06.062

is the blow-up technique, which is explained in Section 2.2. Roughly speaking the idea behind this method is to explode, through a change of variables that is not a diffeomorphism, the singularity to a line or to a circle. Then the study of the original singular point can be reduced to the study of the new singular points that appear on this line or circle and that will be, probably, simpler. If these new singular points are again degenerate the process is repeated.

A vector field X satisfies a *Łojasiewicz inequality* if there exist some $k, c, \delta > 0$ such that $\|X(x, y)\| \geqslant c\|(x, y)\|^k$, for $\|(x, y)\| < \delta$. Dumortier showed in [8] that, for a given singular point of a $\mathcal{C}^\infty$ vector field satisfying a Łojasiewicz inequality (which includes the analytic case), this chain of changes of variables is finite. However, the process of desingularizing a singular point is very long and it involves a big number of computations. There are several generalizations of this technique that consist in doing several blow-ups at the same time, see for instance [9,1,2]. But although they shorten the blow-up process, a previous study of the system has to be done to apply these generalizations and consequently the study of the point is still very tedious and long.

In this work we study the relationship between some integrability objects and the topological behavior of the singular points. Concretely we develop a simple algorithm which allows to completely characterize the topological behavior of the orbits of an analytic system in a neighborhood of a degenerate singular point at the origin, no matter its degeneracy, under the assumption that a generalized rational first integral is defined. This characterization is given in terms of the curves passing through the origin and of their multiplicity. As far as we know, this is the first work in which the first integral is applied to characterize the local behavior of degenerate singular points. In some sense, we blow up the first integral. As a particular case we apply the method when the system is polynomial and has a rational first integral.

The paper is structured as follows. In Section 2 we give the main definitions on analytic functions and integrability, and we explain how the blow-up technique works. In Section 3 we provide some preliminary results that will be necessary for stating the algorithm, which is presented in Section 4. Finally, in Section 5 we show some examples of application. One of them considers the inverse problem of constructing differential system having a first integral and having a given set of curves as separatrices and a given distribution of canonical regions in a previously determined way. A natural question is afterwards raised.

2. Basic definitions and results

2.1. Analytic functions and integrability

We first briefly introduce the notions of formal power series and analytic functions. For more information we refer the reader to the work of Seidenberg (see [14]), see also [15,6]. Let

$$\mathbb{C}[\![x, y]\!] = \left\{\varphi(x, y) = \sum_{i,j} \varphi_{i,j} x^i y^j\colon\ \varphi_{i,j} \in \mathbb{C}\right\}$$

be the ring of formal power series in two variables with coefficients in $\mathbb{C}$. With the usual operations of addition and multiplication, this ring is factorial. The elements of the subring $\mathbb{C}\{x, y\}$ of convergent power series are said to be *analytic* functions.

Let $\varphi(x, y) \in \mathbb{C}[\![x, y]\!] \setminus \{0\}$ be an irreducible non-unit element, i.e. $\varphi(0, 0) = 0$. An *analytic branch* centered at $(0, 0)$ is the equivalence class of φ under the equivalence relation $\varphi \sim \psi$ if $\varphi = \nu\psi$, where ν is a unit element, i.e. $\nu(0, 0) \neq 0$.

A *solution* of a formal differential equation $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ is an analytic branch $\varphi(x, y)$ centered at the origin such that there exists $k(x, y) \in \mathbb{C}\{x, y\}$ satisfying $P\varphi_x + Q\varphi_y = k\varphi$. k is the *cofactor* of φ.

In the following we introduce some notions of integrability. If $U \subseteq \mathbb{R}^2$ is an open set, a non-constant $\mathcal{C}^1$ function $H : U \to \mathbb{R}$, eventually multi-valued, which is constant on all the solutions of X contained in U is a *first integral* of X on U. Moreover we have $XH = 0$ on U. The importance of the first integral is on its level sets: the existence of such a function H determines the phase portrait of the system on U, because the level sets $H = h \in H(U)$ provide the expression of the curves laying on U. Consequently, given a differential system (2), it is important to know whether it has a first integral.

Two analytic functions $f(x, y)$ and $g(x, y)$ defined on a subset $U \subset \mathbb{R}^2$ are said to be *coprime* if the set of points $\{(x, y) \in U\colon\ f(x, y) = g(x, y) = 0\}$ is isolated. We call the ratio of two coprime analytic functions a *generalized rational function*. A generalized rational function $H = f/g$ defined on U is a first integral of system (2) if $\Sigma = \{(x, y) \in U\colon\ g(x, y) = 0\}$ is a set of integral curves of system (2) and H is a first integral on $U \setminus \Sigma$. Obviously, $H = f/g$ is a first integral of system (2) if and only if $(Xf)g - (Xg)f = 0$ on U.

The following theorem (see [12]) is an extension of the Poincaré Theorem (see [13]) for generalized rational first integrals.

Theorem 1. *Assume that the origin is an elementary singular point of the analytic differential system*

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \cdots \tag{3}$$

with eigenvalues $\lambda_1 \neq 0$ and λ_2. Then system (3) *has a generalized rational first integral in some neighborhood of the origin if and only if one of the following conditions holds*:

(i) $\lambda_1 \neq 0 = \lambda_2$ *and the origin is not an isolated singular point*;
(ii) $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \in \mathbb{Q}^+ \setminus \mathbb{N}$;
(iii) $\lambda_1 = \lambda_2 \neq 0$, $A = \mathrm{diag}(\lambda_1, \lambda_2)$;
(iv) $\lambda_1/\lambda_2 \in \mathbb{N} \setminus \{1\}$ *or* $\lambda_2/\lambda_1 \in \mathbb{N} \setminus \{1\}$ *and the germ* (3) *is analytically equivalent to its linear part*;
(v) $\lambda_1/\lambda_2 \in \mathbb{Q}^-$ *and the germ* (3) *is analytically orbitally equivalent to its linear part.*

From this theorem we know that if an analytic system has a generalized rational first integral then any elementary singular point must be either a saddle, or a center, or a node; it cannot be neither a focus nor a saddle-node.

The notion of remarkable curve of a rational first integral was introduced by Poincaré (see [13]). It is proved in [5] that there are finitely many remarkable values for a given rational first integral. As far as we know, since Poincaré's ones very few results have been published about the remarkable values with the exception of these last years (see [5] and [11]).

We next introduce the notions of remarkable values and remarkable curves for generalized rational first integrals. We say that $c \in \mathbb{C}$ (resp. $c = \infty$) is a *remarkable* value of a generalized rational first integral $H = f/g$ if $f + cg = f_1^{n_1} \cdots f_r^{n_r}$ (resp. $g = f_1^{n_1} \cdots f_r^{n_r}$), where $r, n_i \in \mathbb{N}$ and $f_i \in \mathbb{C}\{x, y\}$ are non-constant solutions of system (2), for all $i \in \{1, \dots, r\}$. The curves $f_i = 0$ are called *remarkable* curves and the n_i are their *exponents*. If some exponent n_i is bigger than one, then c and f_i are said to be *critical*. Finally we define the *remarkable factor* R to be the product of all the critical remarkable curves of H powered to their corresponding exponent minus one. We note that the function R is the greatest common divisor of $g^2 H_x$ and $g^2 H_y$.

Next proposition improves a result of [11] about the relationship between the exponents of two curves passing through an elementary singular point and its behavior.

Proposition 2. *Assume that the differential system* (2) *has a generalized rational first integral* $H = f/g$. *Suppose that the origin is an elementary singular point and that exactly two branches* (*real or complex*) *of the curve* $fg = 0$ *cross it. Suppose that the two branches correspond to irreducible solutions* $f_1 = 0$ *and* $f_2 = 0$, *and let* $n_1, n_2 \in \mathbb{Z} \setminus \{0\}$ *be their respective exponents in the expression of* H. *Then*:

(i) *If* $n_1 n_2 < 0$ *then the origin is a node.*
(ii) *If* $n_1 n_2 > 0$ *and both branches are real, then it is a saddle.*
(iii) *If* $n_1 n_2 > 0$ *and both branches are complex conjugate, then it is a center.*

Moreover, the quotient of the eigenvalues at the origin is a positive rational multiple of $-n_1/n_2$.

Proof. Set $H = \prod_{i=1}^{p} f_i^{n_i}$ and let k_i be the cofactor of f_i, for all i. Applying $Xf_i = k_i f_i$ at the origin, as $f_i(0,0) \neq 0$ for $i > 2$ and $(0,0)$ is a singular point of the system, one obtains $k_i(0,0) = 0$ for all $i > 2$.

As $XH = 0$, applying this expression at the origin and after straightforward computations, we have $\sum_{i=1}^{p} n_i k_i(0,0) = 0$, and then $n_1 k_1(0,0) + n_2 k_2(0,0) = 0$. Applying similar arguments to those of [6], both cofactors at the origin are integer multiples of the two eigenvalues of the vector field at $(0,0)$, say $k_1(0,0) = s_1 \lambda$ and $k_2(0,0) = s_2 \mu$, for $s_1, s_2 \in \mathbb{N}$ and λ, μ the eigenvalues. Hence

$$\frac{s_1}{s_2}\frac{\lambda}{\mu} = \frac{k_1(0,0)}{k_2(0,0)} = -\frac{n_2}{n_1},$$

and therefore the proposition follows. $\square$

2.2. The technique of the blow-up

Consider the real planar analytic differential system

$$\dot{x}_i = P(x_i, y_i), \qquad \dot{y}_i = Q(x_i, y_i) \tag{4}$$

of multiplicity m in the variables (x_i, y_i) and assume that the origin is a degenerate singular point of this system. The *directional blow-up in the* x_i (*resp.* y_i) *direction* is the change of variables $(x_{i+1}, y_{i+1}) = (x_i, y_i/x_i)$ (resp. $(x_{i+1}, y_{i+1}) = (x_i/y_i, y_i)$). This transformation converts the origin of the (x_i, y_i)-plane into the line $x_{i+1} = 0$ (resp. $y_{i+1} = 0$). The expression of system (4) after the blow-up, for instance in the x_i direction, is

$$\begin{aligned} \dot{x}_{i+1} &= P(x_{i+1}, x_{i+1} y_{i+1}), \\ \dot{y}_{i+1} &= \frac{Q(x_{i+1}, x_{i+1} y_{i+1}) - y_{i+1} P(x_{i+1}, x_{i+1} y_{i+1})}{x_{i+1}}, \end{aligned} \tag{5}$$

that is always well defined since we are assuming that the origin is a singularity.

We note that, after the blow-up, x_{i+1}^{m-1} is a common factor of $\dot{x}_{i+1}$ and $\dot{y}_{i+1}$. Thus we scale the independent variable to remove it. Along all this paper, when working with system (5) we will assume that such a reparametrization has been done.

We reproduce in the following two well-known results (see [4]) that provide the relationship between the original singular point of system (4) and the new singularities of system (5). We recall that a *characteristic direction* of system (4) is a solution of the equation

$$\mathcal{F}_m(x_i, y_i) := x_i Q_m(x_i, y_i) - y_i P_m(x_i, y_i) = 0, \tag{6}$$

provided that this polynomial is not identically zero. We call $\mathcal{F}_m$ the *characteristic polynomial*. If $\mathcal{F}_m \not\equiv 0$ and φ_t is a solution of system (4) tending to the origin in forward or backward time, then it must do it tangent to one of the characteristic directions.

Proposition 3. *Let $\varphi_t = (x_i(t), y_i(t))$ be a trajectory tending to the origin of system (4), in forward or backward time. Suppose that $\mathcal{F}_m \not\equiv 0$. Assume that φ_t is tangent to one of the two angle directions $\tan\theta = v$, $v \neq \infty$. Then the following statements hold.*

(i) *The two angle directions $\theta = \arctan v$ (in $[0, 2\pi)$) are characteristic directions.*
(ii) *The point $(0, v)$ on the (x_{i+1}, y_{i+1})-plane is an isolated singular point of system (5).*
(iii) *The trajectory φ_t corresponds to a solution of system (5) tending to the singular point $(0, v)$.*
(iv) *Conversely, any solution of system (5) tending to the singular point $(0, v)$ on the (x_{i+1}, y_{i+1})-plane corresponds to a solution of system (4) tending to the origin in one of the two angle directions $\tan\theta = v$.*

The conclusion of the previous proposition is that in order to study the behavior of the solutions around the origin of system (4) it is enough to study the singular points of the form $(0, v)$ of system (5), that will be simpler. But, as we said before, it is possible that, despite they are simpler, some of them are still quite complicated. If this is the case, then we have to study these degenerate singularities by blowing them up and repeating the process.

If $\mathcal{F}_m \equiv 0$ then it is clear that there exists a homogeneous polynomial of degree $m-1$ $W_{m-1} \not\equiv 0$ such that $P_m = xW_{m-1}$ and $Q_m = yW_{m-1}$. We call the directions satisfying $W_{m-1} = 0$ *singular directions*.

Proposition 4. *If $\mathcal{F}_m \equiv 0$ then for every non-singular direction θ there exists exactly one semipath tending to the origin in the direction θ. If θ^* is a singular direction, there may be either no semipaths tending to the origin in the direction θ^*, or a finite number, or infinitely many.*

3. Preliminary results

In this section we state and prove several results that will be useful in order to show that the algorithm that we state in Section 4 works. All along this section we work with an analytic system of type (4) and its corresponding blown-up system (5) in the x_i direction (the case where the y_i directional blow-up is applied follows in a similar way). We assume that system (4) has a generalized rational first integral $H = f/g$.

We denote by m_h the multiplicity of an analytic function h at the origin. We also assume that for every system only one directional blow-up is needed. This means the following: if we want to do the blow-up for instance in the x_i direction to system (4), then there is no curve tending to the origin in this direction. This can be easily ensured by a convenient rotation of the system.

We use the notation f, g and R also for the corresponding numerator, denominator and remarkable factor of the blown-up first integrals associated to H. In a similar way we use $\mathcal{F}_m$, and we use m to refer to the multiplicity at a considered singular point.

In the whole process of desingularization we denote the variables of the systems as (x_i, y_i), $i \in \mathbb{N} \cup \{0\}$. We start with $(x_0, y_0) = (x, y)$; the $(i+1)$-th blow-up is $x_{i+1} = x_i$, $y_{i+1} = y_i/x_i$, and it goes from the i-th desingularization of the initial system (2) to the $(i+1)$-th one. The results in this section are stated for system (4).

First we remark how the integrability objects are transformed after the $(i+1)$-th blow-up.

Lemma 5. *If $h(x_i, y_i) = 0$ is a solution of system (4) with cofactor k, then the function $h(x_{i+1}, x_{i+1}y_{i+1})/x_{i+1}^{m_h} = 0$ is a solution of system (5) with $k(x_{i+1}, x_{i+1}y_{i+1})/x_{i+1}^{m_k}$ as cofactor. Moreover the functions $H(x_{i+1}, x_{i+1}y_{i+1})$ and $x_{i+1}^{\omega} R|_{y_i = x_{i+1}y_{i+1}}$, where $\omega = |m_f - m_g| - 1 - m_R$, are respectively a first integral and the remarkable factor of system (5).*

The proof of Lemma 5 follows from straightforward computations. Obviously the results also follow when the blow-up $x_i = x_{i+1}y_{i+1}$ is applied instead of $y_i = x_{i+1}y_{i+1}$.

From Lemma 10 below we obtain the multiplicity of R for system (2) in terms of the multiplicities of the system and of the curves f and g. The multiplicities of the remarkable factors of the blown-up systems can be computed using the multiplicity of R and Lemma 5. We note that we do not need the expression of R but its multiplicity, which is computable using Lemma 10.

In the case where H is a rational first integral, all the remarkable curves of H can be computed, as there are in the literature several methods to compute them, for instance the one concerning the extactic curves (see [7,11]) and a new one provided in [10].

The following proposition allows to control whether the characteristic polynomial $\mathcal{F}_m$ of a blown-up system is identically zero without computing the differential system explicitly. We denote by $\hat{h}$ the first non-zero jet of an analytic function h.

Proposition 6. *We have $\mathcal{F}_m \equiv 0$ if and only if $m_{f+cg} = m_g$ for all $c \in \mathbb{C}$.*

Proof. Suppose that there is no $s \in \mathbb{C}$ such that $\hat{f} + s\hat{g} \equiv 0$. Then $f + cg$ has always the same multiplicity and therefore $(\hat{f} + c\hat{g})|_{y_i = x_{i+1} y_{i+1}} / x_{i+1}^{m_g}$ is a polynomial in y_{i+1} with c as a parameter that has, varying c, infinitely many roots. Therefore $\mathcal{F}_m = 0$ has infinitely many roots. As $\mathcal{F}_m$ is to be a polynomial, we have $\mathcal{F}_m \equiv 0$.

On the other side, if $\mathcal{F}_m \equiv 0$ then the origin is crossed by the solutions of the system with infinitely many slopes. Suppose that there exists $s \in \mathbb{C}$ such that $m_{f+sg} \neq m_g$. Assume, without loss of generality, that $s = 0$. Then $\widehat{f + cg} = 0$ is equivalent either to $\hat{g} = 0$ or to $\hat{f} = 0$ for all $c \in \mathbb{C}$ and therefore the number of different slopes is finite, a contradiction. $\square$

Remark 1. If $\mathcal{F}_m \not\equiv 0$ then there exists $s \in \mathbb{C} \cup \{\infty\}$ such that $m_{f+sg} > m_{f+cg}$ for all $c \in \mathbb{C} \cup \{\infty\}$, $c \neq s$.

3.1. The dicritical case

The case $\mathcal{F}_m \equiv 0$ is called the *dicritical case*. We recall that in the dicritical case we can write $P_m = xW_{m-1}$ and $Q_m = yW_{m-1}$, where W_{m-1} is a homogeneous polynomial of degree $m-1$. Moreover, the blown-up system (5) has a line of singularities, all of them semi-hyperbolic except a finite number. This finite set of singular points is a subset of the singular points which correspond to the singular directions of system (4), which is obtained from the equation $W_{m-1} = 0$. Once they are known, they can be studied separately.

The following proposition, due to Maria Alberich and A.F., allows to compute the singular directions in terms of the first integral.

Proposition 7. *Let w be a homogeneous polynomial. Let $e_3 \in \mathbb{N} \cup \{0\}$ be the exponent of w in the factorization of $\hat{R}$. Consider the following property*:

(H1) *There exists $c \in \mathbb{C} \cup \{\infty\}$ such that w is a multiple factor of $\hat{f} + c\hat{g}$ with multiplicity $e_1 \in \mathbb{N} \setminus \{1\}$ and $w^{e_1} \nmid \gcd(\hat{f}, \hat{g})$.*

Then a divisor w of W_{m-1} either satisfies (H1) *or $w | \gcd(\hat{f}, \hat{g})$. Conversely, let w be a homogeneous polynomial such that either* (H1) *holds or w divides $\gcd(\hat{f}, \hat{g})$ with multiplicity $e_2 \in \mathbb{N}$. Then $w^e \mid W_{m-1}$ and $w^{e+1} \nmid W_{m-1}$, where $e = e_1 - 1 + e_2 - e_3$ if* (H1) *holds* (*here $e_2 = 0$ if $w \nmid \gcd(\hat{f}, \hat{g})$*) *and $e = e_2 - e_3$ otherwise.*

Proof. Let $S^{x_i} := g^2 H_{x_i} = f_{x_i} g - f g_{x_i}$ and $S^{y_i} := -g^2 H_{y_i} = -f_{y_i} g + f g_{y_i}$. As H is a first integral of system (2) we have $(P, Q) = (S^{y_i}, S^{x_i})/R$. From the equalities $\widehat{S^{y_i}} = -\hat{f}_{y_i}\hat{g} + \hat{f}\hat{g}_{y_i}$ and $\widehat{S^{x_i}} = \hat{f}_{x_i}\hat{g} - \hat{f}\hat{g}_{x_i}$, it is clear that $\gcd(\hat{f}, \hat{g})$ divides $\widehat{S^{x_i}}$ and $\widehat{S^{y_i}}$.

Now if $c \in \mathbb{C} \cup \{\infty\}$ is such that $\hat{f} + c\hat{g} = 0$ has a multiple factor w, then both $(\hat{f} + c\hat{g})_{x_i}$ and $(\hat{f} + c\hat{g})_{y_i}$ vanish on $w = 0$. Hence on $w = 0$ we have

$$\frac{\hat{f}}{\hat{g}} = \frac{\hat{f}_{y_i}}{\hat{g}_{y_i}} = \frac{\hat{f}_{x_i}}{\hat{g}_{x_i}} = -c,$$

and therefore w divides both $\widehat{S^{x_i}}$ and $\widehat{S^{y_i}}$.

On the other hand, let w be a common factor of $\widehat{S^{x_i}}$ and $\widehat{S^{y_i}}$. Then on $w = 0$ we have

$$\hat{f}_{y_i}\hat{g} = \hat{f}\hat{g}_{y_i}, \qquad \hat{f}_{x_i}\hat{g} = \hat{f}\hat{g}_{x_i}. \tag{7}$$

If w divides $\gcd(\hat{f}, \hat{g})$ with multiplicity $e_2 \in \mathbb{N}$ then these equalities hold on $w = 0$. Moreover we can write (7) as

$$\frac{\hat{f}}{\hat{g}} = \frac{\hat{f}_{y_i}}{\hat{g}_{y_i}} = \frac{\hat{f}_{x_i}}{\hat{g}_{x_i}}.$$

All the polynomials in these equalities are homogeneous and the numerators and denominators have the same degree two by two, hence all the quotients are equal to a constant, say $-c$. Then there exists $e_1 \in \mathbb{N}$, $e_1 > e_2$, such that $w^{e_1} | (\hat{f} + c\hat{g})$.

The expression of e follows from the explanation above and $(P, Q) = (S^{y_i}, S^{x_i})/R$. $\square$

Remark 2. Proposition 7 allows to compute the singular directions without computing the differential system explicitly. Moreover as a consequence of the computation, we construct the polynomial W_{m-1}, and hence the value of m appears naturally, as it is the degree of W_{m-1} plus one.

Proposition 8. *Suppose that the origin is a dicritical singular point corresponding to a singular direction.*

(i) *If $m > 1$ then another blow-up is required.*
(ii) *If $m = 1$ then it is a star-node.*

The study of the case where the origin is not dicritical is done in the next subsection.

3.2. The non-dicritical case

The dicritical case leads either to the non-dicritical one or to a star-node, hence it remains to study the non-dicritical case. From now on in this section we assume that $\mathcal{F}_m \not\equiv 0$.

If $m_f = m_g$ then there exists $s \in \mathbb{C} \cup \{\infty\}$ such that $m_{f+sg} > m_g$ (see Remark 1). This curve factorizes after the blow-up (for instance $y_i = x_{i+1} y_{i+1}$ and after removing $x_{i+1}^{m_g}$) as a positive power of x_{i+1} and another polynomial, say W. Therefore from the intersection of the curves $x_{i+1} = 0$ and $W = 0$ new singular points may appear.

From now on and until the end of Section 4 we assume that the curve $f + sg = 0$ defined in Remark 1 is in the numerator of H, that is $s = 0$. This transformation can be easily done taking $H + s = (f + sg)/g$ as first integral instead of H.

Remark 3. From Lemma 5 we know that $x_{i+1} = 0$ is a remarkable curve of (5) and that the first integral of system (5) has the same remarkable values as H and also $c = 0$, as we are assuming that $m_f > m_g$.

The following proposition ensures that all the orbits that are needed in the desingularization process are contained in the curves appearing in the expression of H.

Proposition 9. *The whole set of characteristic directions of the differential system* (4) *at the origin is obtained from the tangents at the origin of $fg = 0$; i.e. the set of solutions y_i/x_i of the equation $\widehat{fg} = 0$.*

Proof. As $m_f > m_g$, $\widehat{f + cg}$ is equal to $\hat{g}$ for all $c \neq 0$ and to $\hat{f}$ for $c = 0$. Hence all the characteristic directions of all the solutions of the system at the origin are found either in $\hat{f} = 0$ or in $\hat{g} = 0$. □

Remark 4. By Proposition 9 we can compute the singular points on $x_{i+1} = 0$ after the blow-up $y_i = x_{i+1} y_{i+1}$ without computing the differential system explicitly. From Proposition 2, if the singular points are elementary then we can characterize them. Otherwise a new blow-up is required.

Next result allows to compute the multiplicity m of system (4) at the origin. We shall see in Section 4 that the knowledge of m is a key point in the application of our algorithm.

Lemma 10. *We have*

$$m_f + m_g - m_R = m + 1. \tag{8}$$

Proof. We write P and Q in terms of f and g:

$$P = -\frac{f_{y_i} g - f g_{y_i}}{R}, \qquad Q = \frac{f_{x_i} g - f g_{x_i}}{R},$$

where R is the remarkable factor. We have

$$R(x_i Q - y_i P) = \left(x_i \frac{\partial f}{\partial x_i} + y_i \frac{\partial f}{\partial y_i} \right) g - \left(x_i \frac{\partial g}{\partial x_i} + y_i \frac{\partial g}{\partial y_i} \right) f = (m_f - m_g) \hat{f} \hat{g} + \cdots.$$

Therefore the lemma follows directly taking multiplicities in the equalities above, as we are assuming $\mathcal{F}_m \not\equiv 0$ and $m_f \neq m_g$. □

Remark 5. When applying a blow-up to a differential system (4), by means of a change of time we cancel a factor x_{i+1}^{m-1} appearing in both $\dot{x}_{i+1}$ and $\dot{y}_{i+1}$. If m is even then this change of time implies a change in the orientation of the orbits contained in the half-plane $x_{i+1} < 0$. Thus from Proposition 10 in the non-dicritical case and also from Proposition 7 in the dicritical case we can compute m to know if such a change of time is to be done, altogether without computing the system explicitly.

Proposition 11. *Assume that a blow-up $y_i = x_{i+1} y_{i+1}$ is applied to system* (4). *Suppose that the origin is a singular point of system* (5) *with multiplicity m. Then*:

(i) *If* $m > 1$ *then the origin is degenerate.*
(ii) *If* $m = 1$ *and two branches of* $fg = 0$ *pass through the origin not transversally, then it is nilpotent.*
(iii) *If* $m = 1$ *and both* $f = 0$ *and* $g = 0$ *pass through the origin transversally, then it is a node. Moreover* $f + cg = 0$ *crosses the origin of system* (4) *with the slope of* $g = 0$*, for all* $c \in \mathbb{C} \setminus \{0\}$.
(iv) *If* $m = 1$ *and two branches of* $f = 0$ *pass through the origin transversally, then it is either a saddle or a center, depending on whether the branches are real or complex, respectively. In the saddle case these branches form the separatrices of the origin.*

Proof. We prove each subcase separately.

(i) The origin of system (5) is not elementary, so a new blow-up is required.
(ii) As two branches belonging to two different level sets of H meet the origin in the same direction, the point is not hyperbolic. Thus as $m = 1$ and the origin cannot be semi-hyperbolic (see Theorem 1), it is nilpotent.
(iii) The singular point is a node as it is elementary and $f + cg = 0$ crosses it for all $c \in \mathbb{C} \cup \{\infty\}$.
(iv) The singular point is a saddle or a center as both branches belong to the same level set $c = 0$ of H and they cross transversally. □

Remark 6. We remind here a well-known result due to Seidenberg (see [14]): if a differential system has a node at the origin, then there is exactly one branch crossing the origin with a determined slope and there are infinitely many branches crossing the origin with another determined slope. In our case, $f = 0$ contains the first branch and $f + cg = 0$, with $c \neq 0$, contain the rest of the branches.

4. The algorithm

We explain in this section how our algorithm works. First we assume that the systems we deal with have some properties that are stated in the next subsection.

4.1. Assumptions

(i) If $\mathcal{F}_m \not\equiv 0$ then we take $H = f/g$ with $m_f > m_g$.
(ii) We suppose that only one of the directional blow-ups is to be done.

We make these assumptions for all the systems appearing after the different blow-ups. We note that they are not restrictive. They are done for a better understanding of the explanation and the process.

4.2. Statement of the algorithm

We describe each step of the algorithm.

(a) We check whether $\mathcal{F}_m \not\equiv 0$ and $m_f = m_g$. If this is the case, there exists $s \in \mathbb{C}$ such that $m_{f+sg} > m_g$ and we take $f + sg$ as the numerator of H instead of f.
(b) If $\mathcal{F}_m \not\equiv 0$ then we compute the singular points on $x_{i+1} = 0$ (or $y_{i+1} = 0$, depending on the direction of the blow-up) from the curves $f = 0$ (after dropping the factor $x_{i+1}^{m_f - m_g}$) and $g = 0$, see Proposition 9. If $\mathcal{F}_m \equiv 0$ then we follow Proposition 7.
(c) For each singular point of step (b) we compute the multiplicity m of the blown-up system at this point and check whether another blow-up is required. This can be done using Propositions 11 and 7.
(d) No new desingularization is to be done for elementary singular points (meaning saddles, nodes and centers). For the degenerate singular points a new blow-up is required. In this case we check the initial assumptions for the new system and go back to step (a).
(e) The algorithm ends as the chain of blow-ups is finite.

The construction of a table is very useful to follow the desingularization process. In this table each row corresponds to a step of the algorithm, i.e. to a blow-up. Each change of variables is written in the first column. Two columns named SP_f and SP_g show the singular points that we obtain from f and g, respectively. In the dicritical case we write the dicritical points in the cells of both SP_f and SP_g. Three more columns $\hat{f}$, $\hat{g}$ and $\hat{R}$ show the first non-zero jets of f, g and R after the singular point is moved to the origin. We shall write a $\star$ in the cells where no new (relevant) information is to be added. See the examples in Section 5.

When the table is done all the singular points appearing from all the necessary blow-ups have been computed and studied. It is clear in the non-dicritical case that the singular points come from the intersection of $\hat{f} = 0$ and/or $\hat{g} = 0$ with $x_{i+1} = 0$ on each step. We can also know their behavior from the multiplicity m of the system at the points and from f and g, as we stated in Proposition 11. The dicritical case reduces to the non-dicritical one or to a star-node from Proposition 8.

4.3. Construction of the local phase portraits

Once the desingularization process is finished, we need to *go back* to the initial system. We start at the last system of the desingularization, say (x_n, y_n), for some $n \in \mathbb{N}$, which corresponds to the last row of the table. We situate on the (x_n, y_n)-plane all the singular points (say on $x_n = 0$) and the lines crossing the y_n axis at these points corresponding to the curves belonging to f or g that provide these singular points. The local behavior of the system at all these singular points is known: they are saddles, nodes or centers. The plane is then divided into several canonical regions and we know the behavior of the system at all of them.

Next we check if some half-plane must change the orientation (see Remark 5). We also notice that some quadrants are to be swapped (as it happens in the blow up process) in the following way:

- If we are applying the x_i directional blow-up then the second quadrant of the (x_i, y_i) system, $\{x_i < 0, y_i > 0\}$, goes to the fourth quadrant of the (x_{i+1}, y_{i+1}) system, as $x_{i+1} = x_i < 0$ and $y_{i+1} = y_i/x_i < 0$.
- If we are applying the y_i directional blow-up then the third quadrant of the (x_i, y_i) system, $\{x_i < 0, y_i < 0\}$, goes to the fourth quadrant of the (x_{i+1}, y_{i+1}) system, as $y_{i+1} = y_i < 0$ and $x_{i+1} = x_i/y_i > 0$.

Now we change the variables into the previous ones in the desingularization process, (x_{n-1}, y_{n-1}). The curves we drew are transformed into new curves by the change of variables; for instance, a curve $y_n = a + \cdots$, $a \in \mathbb{R}$, writes also $y_{n-1} = ax_{n-1} + \cdots$, as $y_n = y_{n-1}/x_{n-1}$. All the singular points of the (x_n, y_n)-plane on $x_n = 0$ meet now at the origin. The shapes and situation of the canonical regions can also be modified. The axes remain invariant if they appear in the corresponding expression of the first integral.

We repeat this procedure until we obtain the local phase portrait of the initial system at the origin with the initial variables $(x_0, y_0) = (x, y)$, and then we are finished.

Note that, in the dicritical case, all the points on $x_{i+1} = 0$ are singular, and all of them except a finite number are semi-hyperbolic. Thus for each one of them, say $(0, v)$, there exists exactly one curve on the (x_i, y_i)-plane crossing the origin with slope v (see [4]).

Remark 7. The algorithm we have shown allows to completely study the local behavior around a singular point, no matter how degenerate it is, without needing to use the blow-up technique explicitly. We have presented an alternative method to this technique for planar differential systems having a generalized rational first integral which uses the information that is provided by some specific curves crossing the singular points, that are also computed.

5. Examples

We present in this last section some examples in order to illustrate how the algorithm must be applied. In all cases we show the corresponding table of desingularization and a figure with all the different phase portraits that we need to obtain the phase portrait of the initial system.

As we said in the introduction, a particular case of analytic curves are the polynomial ones. The first example deals with a polynomial system having a rational first integral.

Example 1. Consider the rational function $H = f/g$, where $f(x, y) = -(x^6y^3 + x^{10} - 6x^4y^6 - y^{10} + 11x^2y^9 + 6x^{10}y^2 - 8y^{12} - 24x^8y^5 + 24x^6y^8 + 12x^{14}y - 24x^{12}y^4 + 8x^{18})(x^6y^3 - x^{10} - 6x^4y^6 + y^{10} + 13x^2y^9 + 6x^{10}y^2 - 8y^{12} - 24x^8y^5 + 24x^6y^8 + 12x^{14}y - 24x^{12}y^4 + 8x^{18})$ and $g(x, y) = (x^2y - 2y^4 + 2x^6)^6$. We want to study the local behavior of the singular point at the origin of the polynomial differential system associated to H. As f and g have both multiplicity eighteen at the origin, we rename $f + g$ (which has multiplicity twenty) as f. Hence we set $f(x, y) = (x^{10} - y^{10} - x^2y^9)^2$. Moreover as $g = 0$ has a vertical tangent at the origin, we apply the change $x \to x + 3y$ to both functions. Let $x_0 = x$ and $y_0 = y$.

We construct Table 1 as it was explained in Section 4. Three blow-ups are needed to completely desingularize the singular point at the origin. We are using Lemma 5 in each blow-up; it tells us how the blow-up transformation affects objects as the first integral and the remarkable factor. From Table 1 we can study all the singular points appearing in the whole blow-up process:

(1) First blow-up, $x_0 = x_1$, $y_0 = x_1y_1$:
- $(0, 0)$: as $m_R = 6$ and $m_f + m_g = 8$ we have from Lemma 10 that $m = 1$. Moreover both $f = 0$ and $g = 0$ pass through this point and $g = 0$ does it transversally, hence it is a node by Proposition 11.
- $(0, -1/2)$ and $(0, -1/4)$: as $m_R = 2$ and $m_f + m_g = 4$ we have from Lemma 10 that $m = 1$ in both cases. Moreover only $f = 0$ passes through these points; applying Proposition 11 we know that they are saddles.
- $(0, -1/3)$: as $m_R = 6$ and $m_f + m_g = 8$ we have from Lemma 10 that $m = 1$. Moreover, both $f = 0$ and $g = 0$ pass through this point and $g = 0$ does it not transversally, hence by Proposition 11 the point is nilpotent and a new blow-up is required.

Before the second blow-up we move the point $(0, -1/3)$ to the origin. Lemma 10 and Proposition 11 are applied in the sequel in the same way as in the first blow-up.

Table 1
Application of the algorithm in Example 1. The $l_i = 0$ are straight lines crossing the corresponding singular point with neither horizontal nor vertical tangency. In particular $l_3 = 2x_2 + 243y_2$ provides the singular point at $x_3 = -243/2$ of the (x_3, y_3)-system. The expressions of the other straight lines are irrelevant for the presentation of the algorithm; as they are neither horizontal nor vertical they provide an elementary singular point when crossing the straight line $x_{i+1} = 0$ or $y_{i+1} = 0$, depending on the blow-up.

	SP_f	SP_g	$\hat{f}$	$\hat{g}$	$\hat{R}$
$y_0 = x_1 y_1$	$y_1 = -\frac{1}{2}$	$\star$	$x_1^2 l_1^2$	$\star$	$x_1 l_1$
	$y_1 = -\frac{1}{4}$	$\star$	$x_1^2 l_2^2$	$\star$	$x_1 l_2$
	$\star$	$y_1 = 0$	x_1^2	y_1^6	$x_1 y_1^5$
	$\star$	$y_1 = -\frac{1}{3}$	x_1^2	x_1^6	x_1^6
$y_1 \to y_1 - \frac{1}{3}$ $x_1 = x_2 y_2$	$x_2 = 0$ l	$x_2 = 0$	x_2^2	$y_2^4 l_3^6$	$x_2 y_2^3 l_3^5$
$x_2 = x_3 y_3$	$x_3 = 0$	$\star$	x_3^2	y_3^8	$x_3 y_3^7$
	$\star$	$x_3 = -\frac{243}{2}$	$\star$	$y_3^8 l_4^6$	$y_3^7 l_4^5$

(2) Second blow-up, $x_1 = x_2 y_2,\ y_1 = y_2$:
- $(0,0)$: as $m_R = 9$ and $m_f + m_g = 12$ we have $m = 2$, hence a new blow-up is required.

(3) Third blow-up, $x_2 = x_3 y_3,\ y_2 = y_3$ (where we take into account that now $m_f < m_g$ and the roles of f and g are swapped):
- $(0,0)$: as $m_R = 8$ and $m_f + m_g = 10$ we have $m = 1$. Moreover, both $f = 0$ and $g = 0$ pass through this point and $f = 0$ does it transversally, hence it is a node.
- $(-243/2, 0)$: as $m_R = 12$ and $m_f + m_g = 14$ we have $m = 1$. Moreover, only $g = 0$ passes through this point, hence it is a saddle.

Now the desingularization process is done. Next we explain how to get the phase portrait of the initial system to end the process.

(1) After the third blow-up we obtain two singular points on the (x_3, y_3)-plane coming from the intersection of $y_3 = 0$ and the curves $x_3 = 0$ and $x_3 = -243/2 + \mathcal{O}(y_3)$.
(2) Back to the (x_2, y_2)-plane we study the origin. The canonical regions of the (x_3, y_3)-system are modified and we have swapped the third and fourth quadrants of the (x_3, y_3)-plane. The curve $y_2 = 0$ remains invariant, and the others become $x_2 = 0$ and $x_2 = -243y_2/2 + \mathcal{O}(y_2^2)$.
(3) Back to the (x_1, y_1)-plane and after swapping again the third and fourth quadrants, $y_2 = 0$ disappears as solution, $x_2 = 0$ becomes $x_1 = 0$ and $x_2 = -243y_2/2 + \mathcal{O}(y_2^2)$ becomes $x_1 = -243y_1^2/2 + \mathcal{O}(y_1^3)$. After this update we undo the change $y_1 \to y_1 - 1/3$ and the singular point is now at $y_1 = -1/3$. There are three more singular points, as Table 1 shows.
(4) Back to the initial system on the (x_0, y_0)-plane and after swapping the second and third quadrants, $x_1 = 0$ disappears as solution and only some branches of $f = 0$ and $g = 0$ remain as separatrices. $f = 0$ provides an elliptic sector and $g = 0$ a hyperbolic sector.

A diagram of the whole process is shown in Fig. 1.

Next example deals with an analytic system having a generalized rational first integral.

Example 2. Let

$$f(x,y) = 3y^4 + x^2y^3 - 2x^3y^2 + 3x^5y - x^4y^2 + 2x^7 - 2x^5y^2 + xy^6 + 3x^4y^4 + y^9 + x^{12}y^8 + \cdots,$$
$$g(x,y) = x^2 - 2y^2 + x^2y - 3x^3y + xy^3 + 5x^5 - 4x^3y^2 + y^5 + \cdots,$$

where the dots mean higher order terms, be two analytic functions and let $H = f/g$. Consider the analytic differential system associated to H. We apply our algorithm in order to study the local behavior around the singular point at the origin of this system. We construct Table 2, from which we study all the singular points appearing in the whole blow-up process. As $m_f = 4$, $m_g = 2$ and $m = 5$, we have $m_R = 0$ from Proposition 10. Therefore $R = 1$. After the first blow-up $y_0 = x_1 y_1$ the remarkable factor is x_1.

(1) First blow-up, $x_0 = x_1,\ y_0 = x_1 y_1$:
- $(0,0)$: as $m_R = 1$ and $m_f + m_g = 5$ we have $m = 3$, hence a new blow-up is required.
- $(0, \sqrt{2}/2)$ and $(0, -\sqrt{2}/2)$: as $m_R = 1$ and $m_f + m_g = 3$ we have $m = 1$ in both cases. Moreover, both $f = 0$ and $g = 0$ pass through these points and $g = 0$ does it transversally, hence they are nodes.

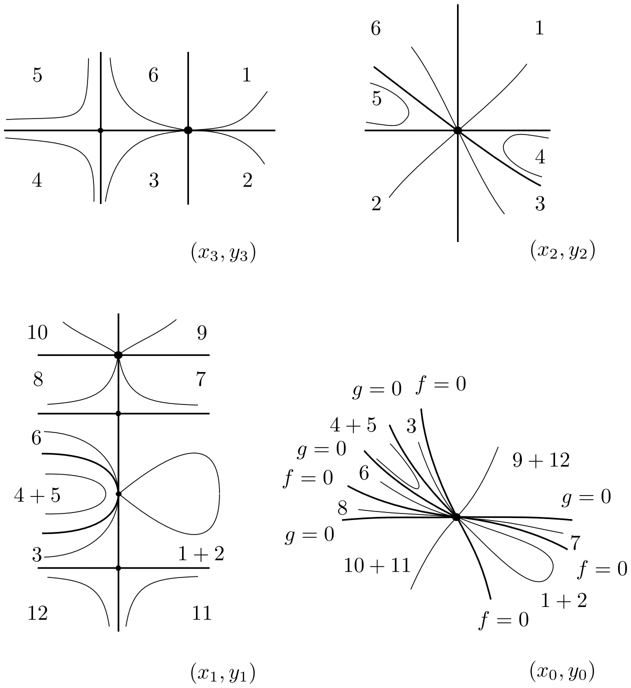

Fig. 1. The desingularization of Example 1. The numbers indicate the canonical regions of the systems to show how they change during the desingularization process. In each system the biggest point represents the origin.

Table 2
Application of the algorithm in Example 2. The $l_i = 0$ are straight lines crossing the corresponding singular point with neither horizontal nor vertical tangency. In particular $l_1 = 2x_1 - y_1$, $l_2 = x_1 + 2y_1$, $l_3 = x_1 - 4y_1$ and $l_4 = 2x_2 - 3y_2$. The expressions of the other straight lines are not relevant.

	SP_f	SP_g	$\hat{f}$	$\hat{g}$	$\hat{R}$
$y_0 = x_1 y_1$	$y_1 = 0$	⋆	$x_1^3 l_1 l_2$	⋆	x_1
	⋆	$y_1 = -\frac{\sqrt{2}}{2}$	x_1^2	l_3	x_1
	⋆	$y_1 = \frac{\sqrt{2}}{2}$	x_1^2	l_3	x_1
$x_1 = x_2 y_2$	$x_2 = 0$	⋆	$x_2^2 y_2^5 l_4$	⋆	$x_2 y_2^4$
	$x_2 = \frac{1}{2}$	⋆	$y_2^5 l_5$	⋆	y_2^4
	$x_2 = -2$	⋆	$y_2^5 l_6$	⋆	y_2^4
$x_2 = x_3 y_3$	$x_3 = 0$	⋆	$x_3^2 y_3^8$	⋆	$x_3 y_3^7$
	$x_3 = \frac{3}{2}$	⋆	$y_3^8 l_7$	⋆	y_3^7

(2) Second blow-up, $x_1 = x_2 y_2$, $y_1 = y_2$:
- $(0, 0)$: as $m_R = 5$ and $m_f + m_g = 8$ we have $m = 2$, hence a new blow-up is required.
- $(1/2, 0)$ and $(-2, 0)$: as $m_R = 4$ and $m_f + m_g = 6$ we have $m = 1$ in both cases. Moreover, only $f = 0$ passes through these points; they are saddles.

(3) Third blow-up, $x_2 = x_3 y_3$, $y_2 = y_3$:
- $(0, 0)$: as $m_R = 8$ and $m_f + m_g = 10$ we have $m = 1$. Moreover, only $f = 0$ passes through this point; it is a saddle.
- $(3/2, 0)$: as $m_R = 7$ and $m_f + m_g = 9$ we have $m = 1$. Moreover, only $f = 0$ passes through this point; it is a saddle.

A diagram of the whole process is shown in Fig. 2. In particular there are two elliptic sectors near the original singular point, see the phase portrait of the (x_0, y_0) system in Fig. 2. Note that quadrants II and III swap in the last desingularization step, and moreover one of the two nodes appearing in the (x_1, y_1) system is an attractor and the other one is a repeller. When shrinking the straight line $x_1 = 0$ to get the initial system (x_0, y_0) there are orbits close to $x_0 = 0$ in the third quadrant approaching (resp. leaving) the origin and orbits in the fourth quadrant leaving (resp. approaching) the origin. Hence the sector is to be elliptic. The same happens with the first and second quadrants.

Something similar happens with regions 7–10, but as all of them are hyperbolic the resulting sectors remain hyperbolic.

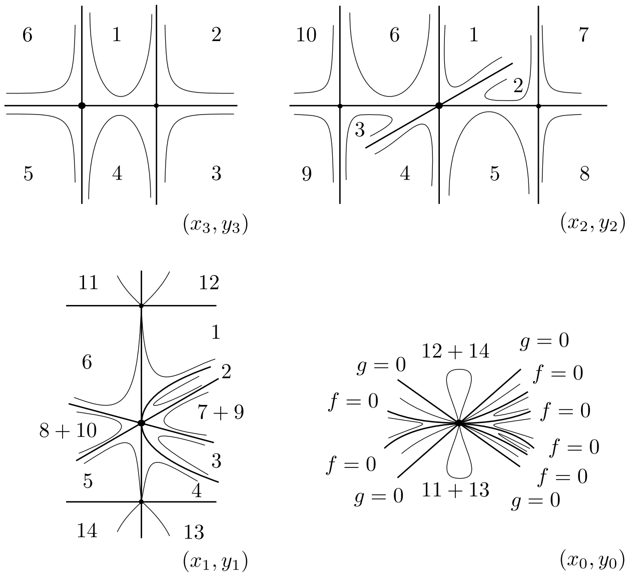

Fig. 2. The desingularization of Example 2.

Table 3
Application of the algorithm in Example 3.

	SP_f	SP_g	$\hat{f}$	$\hat{g}$	$\hat{R}$
$\mathcal{F}_m \equiv 0$	(0, 0)		y_0^2	$(x_0 + y_0)^2$	$x_0 + y_0$
$y_0 = x_1 y_1$	$y_1 = 0$	$\star$	$x_1^2 + y_1^2$	$\star$	1

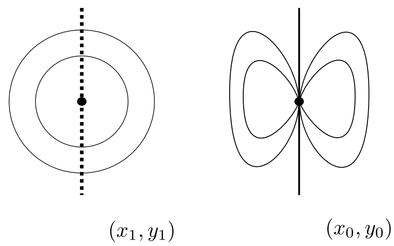

Fig. 3. The desingularization of Example 3.

The following example appears in [4]. It deals with the dicritical case.

Example 3. Let $f(x, y) = y^2 + (x + y)^4$ and $g(x, y) = (x + y)^2$ be two polynomials and let $H = f/g$. Consider the polynomial system associated to H. The polynomials f and g have the same multiplicity at the origin, but there is no $s \in \mathbb{C}$ such that $m_{f+sg} > m_g$, hence we are in the dicritical case.

It is clear that $\gcd(\hat{f}, \hat{g}) = 1$. On the other hand, $\widehat{f + cg}$ has the multiple factors y^2 for $c = 0$ and $(x + y)^2$ for $c = \infty$. Moreover $\hat{R} = 2(x + y)$. Thus from Proposition 7 we obtain $W_{m-1} = W_1 = y$. From this computation we know that the differential system associated to H has multiplicity $m = 2$.

We construct as usual a table of desingularization (see Table 3). Only one blow-up is to be done in order to completely know the behavior of the singular point at the origin of the initial system. After the blow-up $x = x_0 = x_1$, $y = y_0 = x_1 y_1$, the multiplicity at the origin is $m = 1$. Only $f = 0$, which is formed by two complex curves, crosses the origin. We have a center. The desingularization process is finished. Fig. 3 shows how we get the phase portrait of the initial system.

To end this section we consider the following problem: given a finite set of analytic curves crossing the origin, $f_1 = 0, \ldots, f_p = 0$, we want to construct a planar differential system having a generalized rational first integral and having these curves as solutions. Moreover we want to be able to fix *a priori* the behavior (elliptic, hyperbolic or parabolic) of the canonical regions defined by the curves when they meet at the origin.

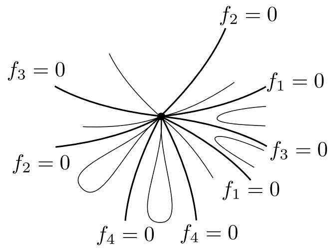

Fig. 4. The local behavior around the origin of the differential system in Example 4.

Table 4
Application of the algorithm in Example 4. The $l_i = 0$ are straight lines crossing the corresponding singular point with neither horizontal nor vertical tangency. In particular $l_1 = 9x_2 + y_2$, $l_3 = 27\bar{x}_2 - 32\bar{y}_2$ and $l_4 = 243\bar{x}_3 - 256\bar{y}_3$. The powers of the factors of $\hat{g}$ marked with a $\star$ are not relevant for the explanation. The polynomial $\hat{R}$ in the first row is $x_1^{n_0+n_1+2n_3-3}$; in the other cases it can be obtained directly from the factors of $\hat{f}$ and $\hat{g}$ powered to their respective exponent minus one.

	SP_f	SP_g	$\hat{f}$	$\hat{g}$
$y_0 = x_1 y_1$	$y_1 = \frac{1}{2}$	$\star$	$x_1^{n_0+n_1+2n_3}$	$\star$
	$\star$	$y_1 = -1$	$x_1^{n_0}$	$x_1^{n_4}$
	$\star$	$y_1 = 0$	$x_1^{n_0}$	$y_1^{n_2}$
$y_1 \to y_1 - 1$ $x_1 = x_2 y_2$	$x_2 = 0$	$x_2 = 0$	$x_2^{n_0} y_2^{n_0-n_4}$	$l_1^{n_4}$
$x_2 = x_3 y_3$	$x_3 = 0$	$\star$	$x_3^{n_0} y_3^{2n_0-2n_4}$	$\star$
	$\star$	$x_3 = -\frac{1}{9}$	$y_3^{2n_0-2n_4}$	$l_2^{n_4}$
$y_1 \to y_1 + \frac{1}{2}$ $x_1 = \bar{x}_2 \bar{y}_2$	$\bar{x}_2 = 0$	$\star$	$\bar{x}_2^{\star} \bar{y}_2^{\star} l_3^{n_1}$	$\star$
$\bar{x}_2 \to \bar{x}_2 + \bar{y}_2$ $\bar{y}_2 = \bar{x}_3 \bar{y}_3$	$\bar{y}_3 = 0$	$\star$	$\bar{x}_3^{\star} \bar{y}_3^{\star} l_4^{n_3}$	$\star$
	$\bar{y}_3 = -1$	$\star$	$\bar{x}_3^{\star} \bar{y}_3^{\star}$	$\star$
	$\bar{y}_3 = \frac{27}{5}$	$\star$	$\bar{x}_3^{\star} l_5^{n_1}$	$\star$
$\bar{x}_3 = \bar{x}_4 \bar{y}_4$	$\bar{y}_4 = 0$	$\star$	$\bar{x}_4^{\star} \bar{y}_4^{\star}$	$\star$
	$\bar{y}_4 = \frac{256}{243}$	$\star$	$\bar{y}_4^{\star} l_6^{n_3}$	$\star$

To get this differential system we need to choose convenient integers $n_1, \dots, n_p$ and to build a function $H = \prod_{i=1}^{p} f_i^{n_i}$ in such a way that the desired behavior between each pair of curves is obtained. We illustrate this idea with an example.

Example 4. Consider the algebraic curves $f_1(x, y) = y^2 - x^3 = 0$, $f_2(x, y) = -x + y - 2xy^2 = 0$, $f_3(x, y) = y^3 + x^5 = 0$ and $f_4(x, y) = 3x^2 + y^3 - 4x^3 y^4 = 0$. We want to construct a system having a rational first integral such that these four curves determine a local behavior around the origin as in Fig. 4. We note that the singular point is not dicritical. From the figure we know that f_2 and f_4 must be in a different level set than f_1 and f_3 (see the parabolic sectors), therefore we take $H = f/g = (f_1^{n_1} f_3^{n_3})/(f_2^{n_2} f_4^{n_4})$, with $n_i \in \mathbb{N}$. We choose these numerator and denominator because the separatrices of the hyperbolic sectors must belong to the same level set, while those of a parabolic sector must belong to different level sets.

The polynomial differential system having this function as (rational) first integral has multiplicity 7 at the origin no matter the values of the $n_i \in \mathbb{N}$. In order to begin the application of our algorithm, first of all we do the change of variables $x \to x + y$, $y \to x - 2y$, as there are curves approaching the origin tangent to both axes. Let $n_0 := m_f - m_g = 2n_1 - n_2 + 3n_3 - 2n_4 \in \mathbb{Z}$. We take $n_0 \neq 0$ as the singular point is not dicritical. We construct Table 4 as usual.

From the construction of Table 4 we can study all the singular points appearing in the whole blow-up process:

(1) First blow-up, $x_0 = x_1$, $y_0 = x_1 y_1$. As we want the singular point $(0, 0)$ in the (x_1, y_1)-plane to be a node in order to obtain an elliptic sector in region 16 (see Fig. 5), we take $n_0 > 0$.
- $(0, 1/2)$: as $m = 2$, a new blow-up is required.
- $(0, -1)$: as $m = 1$, both $f = 0$ and $g = 0$ pass through this point not transversally (because $n_0 > 0$), a new blow-up is required.
- $(0, 0)$: as $m = 1$, both $f = 0$ and $g = 0$ pass through this point and $g = 0$ does it transversally (because $n_0 > 0$), it is a node.

The second and third blow-ups concern the point $(0, -1)$. We move this point to the origin of the (x_1, y_1)-plane.

(2) Second blow-up, $x_1 = x_2 y_2$, $y_1 = y_2$.
- $(0, 0)$: as $m = 2$, a new blow-up is required.

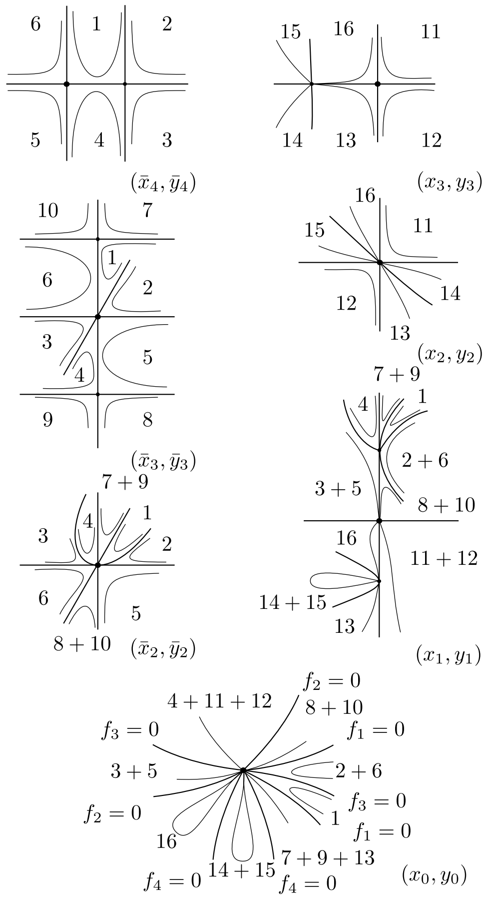

Fig. 5. The desingularization of Example 4.

(3) Third blow-up, $x_2 = x_3 y_3,\ y_2 = y_3$. Because of the configuration that we want to obtain we must take $m_f > m_g$. This implies that we must take $2n_0 > n_4$.
- $(0,0)$: $m = 1$ and only $f = 0$ passes through this point; it is a saddle.
- $(-1/9, 0)$: as $m = 1$, both $f = 0$ and $g = 0$ pass through this point and $g = 0$ does it transversally, it is a node.

The rest of blow-ups concern the point $(0, 1/2)$ in the (x_1, y_1)-plane. We move this point to the origin.

(4) Fourth blow-up, $x_1 = \bar{x}_2\bar{y}_2,\ y_1 = \bar{y}_2$:
- $(0,0)$: as $m = 2$, a new blow-up is required.

Before the next blow-up, and as there are curves with vertical and horizontal tangent, we do the change of variable $\bar{x}_2 \to \bar{x}_2 + \bar{y}_2$.

(5) Fifth blow-up, $\bar{x}_2 = \bar{x}_3,\ \bar{y}_2 = \bar{x}_3\bar{y}_3$:
- $(0,0)$: as $m = 2$, a new blow-up is required.
- $(0,-1)$ and $(0, 27/5)$: in both cases $m = 1$ and only $f = 0$ passes through these points; they are both saddles.

(6) Sixth blow-up, $\bar{x}_3 = \bar{x}_4\bar{y}_4,\ \bar{y}_3 = \bar{y}_4$:
- $(0,0)$ and $(256/243, 0)$: $m = 1$ and only $f = 0$ passes through these points; they are both saddles.

Now the desingularization process is done. To obtain the phase portrait of the initial system we must undo the changes of variables. As a conclusion, in order to ensure that we obtain the desired configuration, we must take $2n_0 > n_4$. A rational first integral is the function $H = (f_1^2 f_3^4)/(f_2 f_4^4)$.

From Example 4 a natural question arises:

Open question. Given a finite set of analytic curves crossing the origin and a local topological configuration around this point, is it possible to find an analytic system having a generalized rational first integral and having a singular point at the origin with the given local topological behavior?

Acknowledgment

The second author wants to thank Maria Alberich many comments that helped us to write this paper.

References

[1] M.J. Álvarez, Critical points and periodic orbits of planar differential equations, PhD thesis, Universitat Autònoma de Barcelona, 2006.
[2] M.J. Álvarez, A. Ferragut, X. Jarque, A survey on the blow up technique, preprint.
[3] A. Andreev, Investigation on the behaviour of the integral curves of a system of two differential equations in the neighborhood of a singular point, Trans. Amer. Math. Soc. 8 (1958) 187–207.
[4] A.A. Andronov, E.A. Leontovich, I.I. Gordon, A.G. Maier, Qualitative Theory of Second Order Differential Equations, John Wiley and Sons, New York, 1973.
[5] J. Chavarriga, H. Giacomini, J. Giné, J. Llibre, Darboux integrability and the inverse integrating factor, J. Differential Equations 194 (2003) 116–139.
[6] J. Chavarriga, H. Giacomini, M. Grau, Necessary conditions for the existence of invariant algebraic curves for planar polynomial systems, Bull. Sci. Math. 129 (2005) 99–126.
[7] C. Christopher, J. Llibre, J.V. Pereira, Multiplicity of invariant algebraic curves in polynomial vector fields, Pacific J. Math. 229 (2007) 63–117.
[8] F. Dumortier, Singularities of vector fields on the plane, J. Differential Equations 23 (1977) 53–106.
[9] F. Dumortier, C. Herssens, Tracing phase portraits of planar polynomial vector fields with detailed analysis of the singularities, Qual. Theory Dyn. Syst. 1 (1999) 97–130.
[10] A. Ferragut, H. Giacomini, A new algorithm for finding rational first integrals of polynomial vector fields, Qual. Theory Dyn. Syst. 9 (2010) 89–99.
[11] A. Ferragut, J. Llibre, On the remarkable values of the rational first integrals of polynomial vector fields, J. Differential Equations 241 (2007) 399–417.
[12] Weigu Li, J. Llibre, Xiang Zhang, Planar analytic vector fields with generalized rational first integrals, Bull. Sci. Math. 125 (2001) 341–361.
[13] H. Poincaré, Sur l'intégration des équations différentielles du premier ordre et du premier degré I and II, Rend. Circ. Mat. Palermo 5 (1891) 161–191, Rend. Circ. Mat. Palermo 11 (1897) 193–239.
[14] A. Seidenberg, Reduction of singularities of the differential equation $A\,dy = B\,dx$, Amer. J. Math. 90 (1968) 248–269.
[15] S. Walcher, On the Poincaré problem, J. Differential Equations 166 (2000) 51–78.